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Internal failures in model elastomeric composites

A. N. GENT, Y.-C. HWANG
College of Polymer Science and Polymer Engineering, The University of Akron, Akron, Ohio 44325-3909, USA

Finite element methods have been used to calculate the rate of release of strain energy caused by growth of an internal crack in some model elastic composites under tension. A layer of a linearly elastic material was considered, bonded between two flat or two spherical rigid surfaces. The reduction in strain energy caused by a small circular crack at the interface was found to be only about one-half of that due to a similar crack in the centre of the layer, in accord with the conjecture of Andrews and King. Cracks in the centre of a thin layer bonded between flat surfaces caused about the same release of energy as a crack in the centre of a thick specimen under the same tensile stress. On the other hand, a crack in a thin layer bonded between two spherical surfaces caused a much larger rate of energy release, depending on the radius of the layer relative to its minimum thickness. Growth of an initial crack would thus occur at a small applied stress. For thin layers between both flat and spherical surfaces, the rate of release of energy decreased as the crack grew, indicating that the crack would stabilize at a finite size. These conclusions are in accord with some observations of cracks in thin elastic layers.

1. Introduction

Cracks grow when there is enough mechanical energy available in the system to drive them forward. This is the Griffith fracture criterion: that energy released by crack growth must be sufficient to meet the energy requirements of a growing crack, termed the fracture energy of the material and denoted here G_c [1, 2]. We have calculated the rate of release of strain energy for a circular crack, of radius c, growing in a layer of an elastic material bonded between two rigid spheres, Fig. 1, or two rigid flat surfaces, Fig. 2. The crack is placed either in the centre of the elastic layer, Figs 1a and 2a, or at the centre of the interface between one rigid material and the elastic layer, Figs 1b and 2b. The first corresponds to an internal crack in the elastomeric material and the second to a defect in adhesion. The corresponding measures of strength are G_c units of energy required to tear through unit area of material and G_a units of energy for debonding unit area of interface.

The elastic material is assumed to be linearly elastic and virtually incompressible. Finite element methods are used to calculate the stiffness of the models for various sizes of the crack, and hence the strain energy, W, corresponding to a given applied force and deflection. In this way the reduction, ΔW, in strain energy brought about by the presence of the crack is evaluated for various crack radii.

A crack will grow if the rate of reduction in strain energy at constant deflection is sufficiently large, i.e. if

$$\partial(\Delta W)/\partial c \geqslant 2\pi c G_c \text{ (or } 2\pi c G_a) \qquad (1)$$

We have evaluated the quantity on the left-hand side of Equation 1 numerically, for a wide range of geometrical shapes. The results are presented here. They enable us to calculate the critical loads at which cracks of a given size will grow, when the fracture energy, G_c or G_a, is known. Some conclusions are also drawn on the final size of cracks formed between two rigid surfaces.

Cracks are initiated in two ways. They occur naturally, as defects within the material or at the bonded interface. Measurements of the strength of rubbery materials suggest that "natural" flaws or stress-raisers equivalent to sharp cracks, about 50 μm in size, are always present [3, 4]. Cracks are also formed within an elastomer by internal fracture under a dilatant stress [5]. Any small void within an elastomeric solid will expand elastically without limit when a critical level of triaxial tension, $-P$, is applied, of about $5E/6$, where E is the tensile (Young's) modulus of elasticity [5, 6]. In practice, the void bursts open to form an internal crack when

$$-P \geqslant 5E/6 \qquad (2)$$

This critical condition is readily set up in elastomeric composites near rigid boundaries. For example, cracks appear abruptly near the poles of an isolated rigid spherical inclusion, in the direction of applied tension, when the local triaxial tension reaches the critical value [7, 8]. When two rigid spheres are located close together in the direction of an applied tension, a crack appears in the elastomer layer midway between them when the critical condition is reached there [8, 9]. Indeed, it seems that a crack always forms where, and when, the critical dilatant stress is set up.

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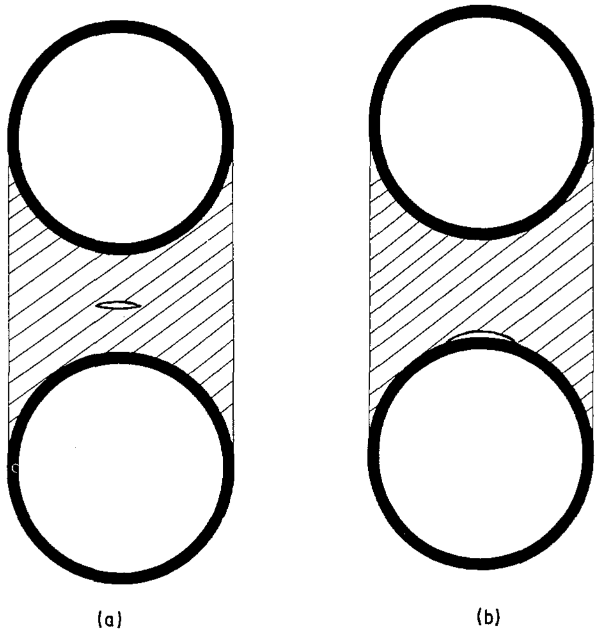

Figure 1 (a) A centre crack, and (b) an interfacial crack in an elastic layer bonded between rigid spherical end-pieces.

We now turn to the question of the applied stress at which cracks will grow and the size that they will eventually attain. These questions are independent of the origin of the cracks themselves, but in considering them we also are led to consider the question of which criterion is met first; Equation 1, for growth of an initial defect, or Equation 2, for bursting open of an initial void.

2. Analytical procedures

A finite element arrangement with cylindrical symmetry was employed, using 400 elements. It is shown schematically for a centre crack of radius c in Figs 3 and 4. In this case only one-half of the complete unit was modelled, but for a single interfacial crack at one surface it was necessary to model the complete unit. Calculations were carried out using the ADINA code [10], the material between the end-pieces being assumed to be linearly elastic with a value of Poisson's ratio, ν, of either 0.4999 or 0.49, corresponding to extreme values for rubber compounds.

Values of applied force, F, were computed for unit deflection of the model and hence the elastic strain energy, W, was obtained, given by $F/2$. These values were smaller, of course, than the value W_0 when no crack was present, and they decreased as the radius c of the crack was made larger, becoming zero when the radius of a central crack reached the radius a of the specimen or when the interfacial crack became equal in size to the original bonded area.

TABLE I Values of l for various thicknesses h of an elastic layer bonded between two spherical or two flat surfaces. l_i and l_c denote values for interfacial and centre cracks, respectively

End pieces	a/h	l_i, $\nu = 0.4999$	l_i, $\nu = 0.49$	l_c, $\nu = 0.4999$	l_c, $\nu = 0.49$
Spherical	50	0.032	–	0.026	–
	10	0.16	–	0.12	–
	5	0.30	–	0.22	–
	1	0.98	0.98	0.75	0.73
	0.5	1.31	1.36	1.06	1.15
	0.1	2.41	2.31	2.15	2.00
Flat	50	0.028	–	0.022	–
	10	0.104	–	0.085	–
	5	0.21	–	0.16	–
	1	0.89	0.97	0.67	0.67
	0.5	1.53	1.60	1.00	1.02
	0.1	–	2.37	2.10	2.06

Values of the reduction, ΔW, relative to the value W_0 in the absence of a crack, are plotted in Figs 5 to 8 as a function of the crack radius c, relative to the radius a of the specimen. Four representative cases are shown: thin and thick elastic layers bonded between spherical end-pieces (Figs 5 and 6), and thin and thick elastic layers bonded between flat end-pieces (Figs 7 and 8).

3. Results and discussion

3.1. Small cracks in thin elastic layers

When the crack was extremely small in comparison with the radius a of the specimen, then the reduction ΔW in strain energy that it caused was too small to determine with any accuracy. As the value of c was increased, a linear relation was found to hold between $\log \Delta W$ and $\log c$, as can be seen in Figs 5 to 8, with a slope of 3 in this representation. Thus, when the crack size was small in comparison with the specimen radius

$$\Delta W/W_0 = kc^3 = (c/l)^3 \qquad (3)$$

where $l\,(= k^{-1/3})$ is a characteristic length of the stress distribution in elastic layers which may be regarded as an inverse measure of the sensitivity of the stress distribution to the presence of a crack. Large values of l correspond to small reductions in elastic strain energy for a crack of given size.

Values of l determined from relations like those shown in Figs 5 to 8 are given in Table I for various thicknesses, h, of the elastic layer. They were found to be virtually the same for the two values of Poisson's ratio used here, 0.49 and 0.4999. No distinction is made hereafter between the two results.

Values of l are plotted against the thickness h, relative to the radius a of the cylindrical specimen, in

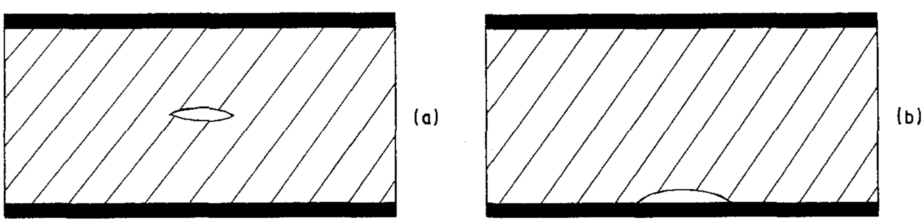

Figure 2 (a) A centre crack, and (b) an interfacial crack in an elastic layer bonded between rigid flat plates.

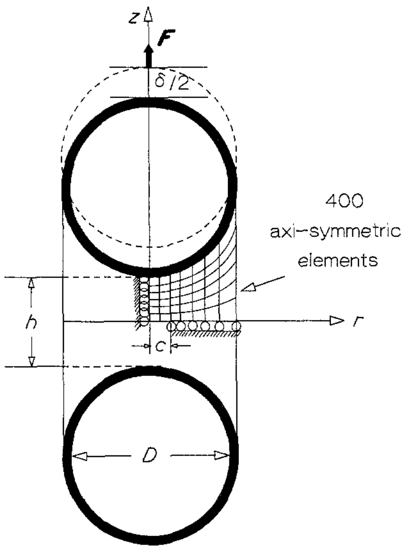

Figure 3 Sketch of finite element arrangements for an elastic layer containing a centre crack, bonded between two rigid spheres.

Figs 9 and 10, using logarithmic scales for both axes. In this representation they follow linear relations initially, with a slope of unity, corresponding to a direct proportionality between l and h

$$l = \alpha h \tag{4}$$

Values of the constant of proportionality α are given in Table II.

They were close to unity in all cases, indicating that the characteristic length l of the stress distribution in thin bonded layers is similar in magnitude to the thickness h of the layer itself. However, they were clearly smaller for cracks growing in the centre of the elastic layer than for interfacial cracks of the same size. Thus, from Equation 3, more energy is released by a central crack than by an interfacial crack. From the computed values of l, we deduce that about twice as much energy is released by a central crack, Table I. This is consistent with the conclusion of Andrews and King [11], that the rate of release of strain energy near a rigid boundary is only one-half of that for a central crack because only one-half as much material is made stress-free as the crack grows.

3.2. Small cracks in thick elastic layers

When the layer thickness, h, was relatively large, of the same order as the radius a of the specimen or larger, then the characteristic length, l, no longer followed a direct proportionality with h. Instead, it tended to increase more slowly, as shown in Figs 9 and 10. The logarithmic relations shown there at large values of h have slopes of 1/3, corresponding to

$$l = \beta h^{1/3} \tag{5}$$

TABLE II Coefficient, α, of the relationship, $l = \alpha h$, for thin bonded layers. α_i and α_c denote values for interfacial and centre cracks, respectively

	α_i	α_c
Spherical surfaces	1.58	1.26
Flat surfaces	1.07	0.87

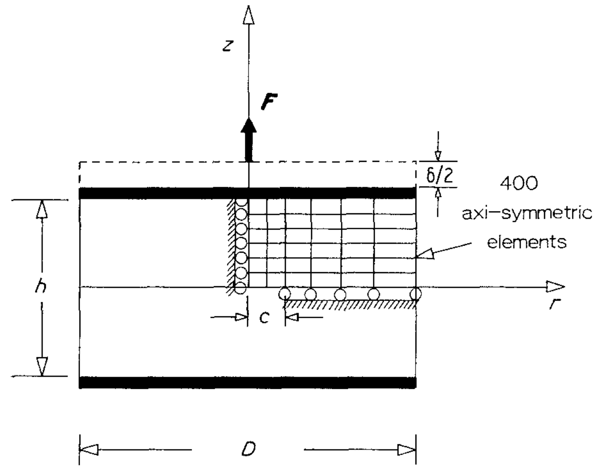

Figure 4 Sketch of finite element arrangements for an elastic layer containing a centre crack, bonded between two rigid plates.

The coefficients β were found to be in satisfactory agreement with theoretical values, derived below, of 0.92 for a centre crack and 1.13 for an interfacial crack.

3.3. Theoretical result for a small crack in a thick layer

Sack's solution for the breaking stress, σ_b, of a long cylindrical specimen containing a small central crack of radius c takes the form (12)

$$\sigma_b^2 = \pi E G_c / 3c \tag{6}$$

where E is the tensile (Young's) modulus of elasticity of the material. Substituting in terms of the strain energy, W_0, and strain energy density, U, i.e. the strain energy per unit volume in regions remote from the crack, where

$$U = \sigma_b^2 / 2E$$

and

$$W_0 = \pi a^2 h (\sigma_b^2 / 2E)$$

and employing the Griffith criterion for propagation of a circular crack of radius c, Equation 1, we obtain

$$\Delta W = 4 c^3 U \tag{7}$$

corresponding to

$$\Delta W / W_0 = (4/\pi) c^3 / a^2 h \tag{8}$$

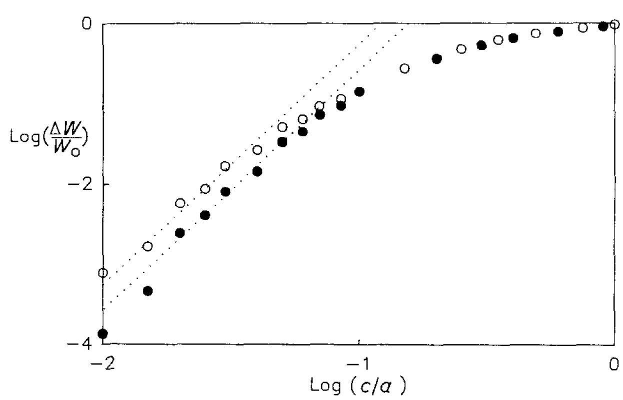

Figure 5 Computed values of reduction ΔW in original elastic energy W_0 due to the presence of a crack of radius c. (●) Interfacial crack; (○) centre crack. Spherical end-pieces, radius a and separation h; $h/a = 0.1$. (····) $\Delta W = kc^3$.

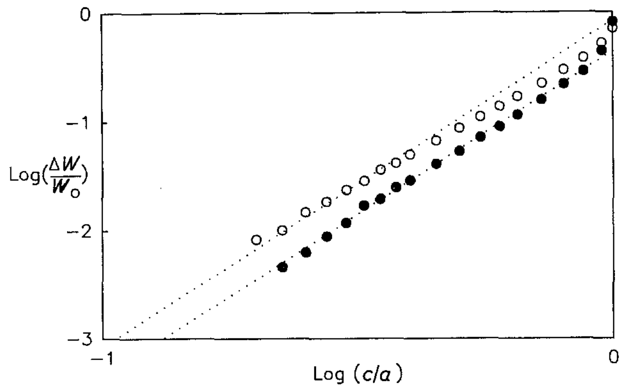

Figure 6 Computed values of reduction ΔW in original elastic energy W_0 due to the presence of a crack of radius c. (●) Interfacial crack; (○) centre crack. Spherical end-pieces, radius a and separation h; $h/a = 2$. (⋯) $\Delta W = kc^3$.

Thus, a small crack in the centre of a long cylindrical block in tension causes a reduction in strain energy given by Equations 7 and 8. On comparing Equations 3 and 8, the characteristic length l is given by

$$l = (\pi a^2/4)^{1/3} h^{1/3} \quad (9)$$

Analogous relations for an interfacial crack take the form

$$\bar{\sigma}_b^2 = 2\pi E G_a / 3c$$

$$\Delta W = 2c^3 U$$

and

$$\Delta W / W_0 = (2/\pi) c^3 / a^2 h$$

in place of Equations 6, 7 and 8.

Thus, the observed form of the dependence of l upon h for thick layers is accounted for, and a theoretical value obtained from Equations 5 and 9 for the coefficient β $(= (\pi a^2/4)^{1/3}$ for a central crack and $(\pi a^2/2)^{1/3}$ for an interfacial crack). A quantitative comparison of these values of β with the calculated results is made in Figs 9 and 10. Values of β for cracks in thick elastic layers are seen to be in satisfactory agreement with the theoretical values when h/a is greater than unity, for specimens with either spherical or flat end-pieces, containing either central or interfacial

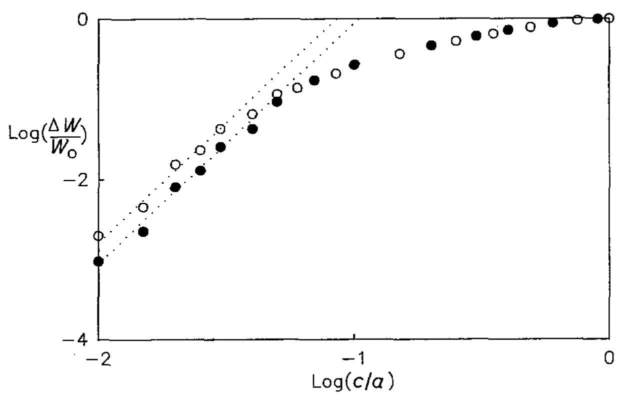

Figure 7 Computed values of reduction ΔW in original elastic energy W_0 due to the presence of a crack of radius c. (●) Interfacial crack; (○) centre crack. Flat end-pieces, radius a and separation h; $h/a = 0.1$. (⋯) $\Delta W = kc^3$.

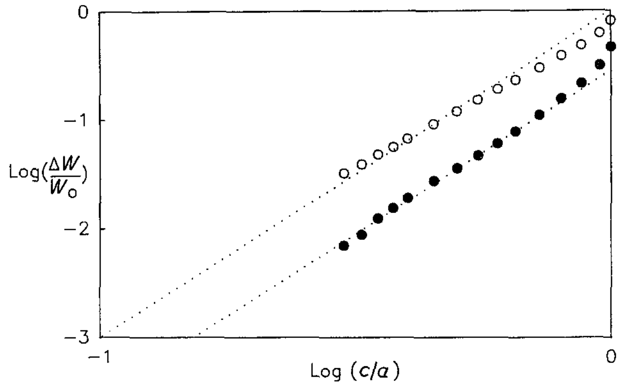

Figure 8 Computed values of reduction ΔW in original elastic energy W_0 due to the presence of a crack of radius c. (●) Interfacial crack; (○) centre crack. Flat end-pieces, radius a and separation h; $h/a = 2$. (⋯) $\Delta W = kc^3$.

cracks. Thus, both the form and magnitude of the computed rate of release of strain energy by a small crack in a thick elastic layer are in reasonable agreement with analytical solutions. This agreement lends support to the other results, when complete analytical solutions are not available.

3.4. Large cracks

The computed relations for reduction ΔW in strain energy, Figs 5 to 8, show interesting differences as the crack radius c is made larger. They depart from a proportionality to c^3, but deviate in different ways, depending upon the layer thickness, h. For relatively thin layers, Figs 5 and 7, they become much less sensitive to crack size, approaching a constant value, i.e. becoming largely independent of c, as c approaches its maximum possible value, the radius a of the cylindrical specimen. For thick elastic layers, on the other hand, Figs 6 and 8, the rate of release of strain energy by a growing crack stays constant or increases when the crack radius becomes large. These differences suggest that a crack growing in a thin layer will slow down and stop, because the rate of release of strain energy becomes less, whereas a similar crack growing in a thick layer will accelerate, in view of the increasing rate at which energy becomes available to it.

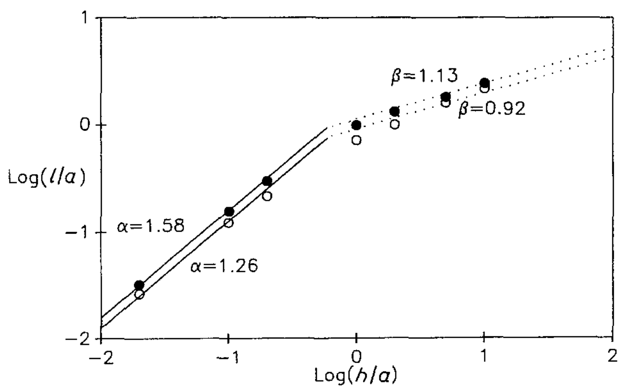

Figure 9 Scaling parameter l for small cracks in an elastic layer bonded between rigid spherical end-pieces, obtained from initial linear relations like those shown in Figs 5 and 6, plotted against the relative thickness h/a of the elastic layer. (●) Interfacial cracks; (○) centre cracks. (——) $l = \alpha h$, (⋯) $l = \beta h^{1/3}$.

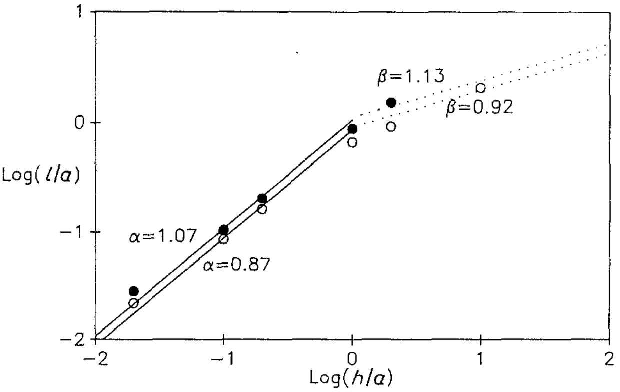

Figure 10 Scaling parameter l for small cracks in an elastic layer bonded between two rigid flat end-pieces, obtained from initial linear relations like those shown in Figs 7 and 8, plotted against the relative thickness h/a of the elastic layer. (●) Interfacial cracks; (○) centre cracks. (——) $l = \alpha h$, (· · · ·) $l = \beta h^{1/3}$.

3.5. Predicted loads at which a small initial crack in a thin elastic layer will grow

Equations 3 and 4 lead directly to a condition for growth of an initial crack of radius c in terms of the strain energy W_0

$$W_0 \geqslant (2\pi/3)\alpha^3 h^3 G_c/c \tag{10}$$

using the Griffith fracture criterion, Equation 1. Now, approximate relations are available for the stiffness, and hence strain energy W_0, of thin bonded elastic layers. For example, for a layer bonded between two flat plates, with a radius a much larger than the thickness h, we have [13, 14]

$$F = \pi a^4 E\delta/2h^3 \tag{11}$$

and for a thin layer bonded between two rigid spheres [15, 16]

$$F = \pi a^2 E\delta/2h \tag{12}$$

On substituting for W_0 in Equation 10, critical values for the mean applied stress $\bar{\sigma}$ $(= F/\pi a^2)$, denoted $\bar{\sigma}_c$, are obtained as

$$\bar{\sigma}_c^2 \geqslant 2\alpha^3 E G_c/3c \tag{13}$$

and

$$\bar{\sigma}_c^2 \geqslant 2\alpha^3 (h/a)^2 E G_c/3c \tag{14}$$

respectively.

Recalling that the coefficient α is approximately equal to unity, Equation 13 indicates that a small crack within a thin bonded layer will grow at a mean applied tensile stress of about the same magnitude as that for a large sample containing a crack of the same size, Equation 6. There is little effect of proximity of bonded planes on the tendency of a crack to propagate. But Equation 14 shows that a crack in an elastic layer bonded between two closely spaced rigid spheres is much more likely to grow. In this case, the critical stress is reduced by the ratio h/a of sphere spacing to radius. For example, if the spacing h is one-tenth of the radius a, then the fracture stress will be only one-tenth of the regular tensile breaking stress.

However, for closely spaced spheres, the rate of release of strain energy falls off markedly as the crack grows, Fig. 5. Thus, although a crack will start to grow at a low stress, it will not continue to propagate until the sample is severed. Instead, it will stabilize at a finite size. This is precisely what is observed [9].

3.6. Crack growth or void expansion?

Equation 13 applies to a pre-existing crack in a thin layer bonded between flat surfaces. Unless the crack is unusually large, it predicts a much greater critical stress than for unbounded expansion of a pre-existing void by a dilatant stress, Equation 2. For example, if E is given a value of 2 MPa, representative of soft elastomeric solids, and G_c is given a value of 1 kJ m^{-2}, typical of reasonably strong rubbery solids, then the fracture stress is calculated from Equation 13 to be about 7.5 MPa, when the initial crack radius is assumed to be 25 μm and putting $\alpha = 1$. On the other hand, the mean applied stress at which a critical dilatant stress of $5E/12$ is reached in the centre is only about 0.9 MPa. Thus, void expansion is likely to be the first mechanism of internal fracture encountered in stretching thin bonded layers, unless they contained exceptionally large initial cracks.

For a thin layer bonded between spherical surfaces, the critical stress for crack growth is much smaller, by the factor h/a, Equation 14. Previous analyses have shown that the maximum dilatant stress $-P_m$ set up in the centre of a thin layer is increased in inverse proportion, relative to the mean applied stress [15, 16]

$$-P_m/\bar{\sigma} = a/h$$

so that the critical stress, $\bar{\sigma}_c$, for void expansion will be reduced by the same factor. Thus, the relative tendency for growth of an initial crack compared to expansion of an existing void is not changed. Both processes are made easier, and by the same factor, in a thin layer bonded between spherical surfaces. Again, therefore, void expansion is likely to be the first failure encountered.

In the above discussion, failure by debonding at the interface has been ignored. As shown previously, stresses for interfacial failure will be higher than for growth of a central crack if the fracture energies are similar. Thus, only if the interface is much weaker than the material itself (or if the interface contains unusually large debonds) will debonding occur before void formation.

4. Conclusions

1. Griffith's fracture criterion for growth of a circular crack of radius c is given in Equation 1. For small cracks in thin bonded layers, the left-hand side of this relation, $\partial(\Delta W)/\partial c$, is given approximately by $3W_0c^2/h^3$, where h is the layer thickness, i.e. the minimum distance separating the rigid bonded surfaces. For small cracks in thick layers this term is given approximately by $3W_0c^2/a^2h$, where a is the radius of the layer. Thus, the effective volume of the specimen, from which energy is released by crack growth, is given approximately by h^3 in the first case and by the volume of the entire layer $(\pi a^2 h)$ in the second.

2. In thin layers, the dependence of this term on c becomes much smaller as the crack grows. Thus a

crack will reach a stable size eventually, without causing the specimen to break in two. In thick layers, on the other hand, once the condition for crack growth is met a crack will grow catastrophically.

3. The reduction in strain energy caused by an interfacial crack is only one-half of that caused by a central crack of the same size. Thus, other things being equal, a central crack will grow preferentially.

4. Simple finite element analyses provide useful information about fracture in model systems, like those considered here, that are somewhat too complicated to be amenable to solution in closed form and yet seem sufficiently general to be of wide application.

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